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SOME PROPERTIES OF STRUCTURAL EQUIVALENCE MEASURES DERIVED FROM SOCIOMETRIC CHOICE DATA *

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I discuss and illustrate the extent to which different relation measures and pattern similarity measures can be expected to generate different structural equivalence results. Measures of network relations and pattern similarity are reviewed to establish clear comparisons between structural equivalence measures. Using Monte Carlo sociometric choice data drawn from four strategically designed study populations, alternative relation and pattern similarity measures are combined in a factorial design generating six measures of structural equivalence within each study population. I report the magnitudes of differences between structural equivalence measures within populations, compared across populations. Three conclusions are drawn: (1) There is significant reliability across alternative measures. (2) This reliability increases with the clarity of boundaries between statuses in a study population. (3) The noticeable differences between structural equivalence measures that exist under conditions at all weaker than strong equivalence are principally a function of how relations are measured rather than how relation pattern similarities are measured. Two inferences are drawn for applied network analysis: (1) Structural equivalence should be computed from path distance measures of network relations (however normalized) rather than being computed directly from patterns of binary choice data. (2) Renewed methodological attention should shift from how we measure pattern similarity to how we measure relationships.

1. Introduction

Much of network analysis relies on the use of sociometric choice data to operationalize structural equivalence concepts of social structure. Broadly stated, analysis proceeds through something like the following steps: Choice data are obtained in various ways such as asking people to name their best friends, or the people to whom they turn for advice, or their spouse or live-in equivalent, or their personal contacts in other

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firms, or the people with whom they discuss important matters. The flow of direct and indirect choices between individuals is used to measure the strength of relations in one or more networks. Patterns of these relations within and across networks are then compared to one another with a pattern similarity measure such as a Euclidean distance or a correlation coefficient indicating the extent to which pairs of individuals are structurally equivalent. The pattern similarity data are studied to locate clusters of structurally equivalent individuals, and each cluster is used to define a row in a density table of the original choice data. Social structure is then described with the distribution of choices in the density table and the distribution of individuals in a spatial map of the pattern similarity data.

Buried in these standard steps are variations that make it difficult to compare analytical results across studies. One important variation from study to study is the transformation of choice data into network relations. Another is the measure used to compare two relation patterns for their equivalence. [1] General purpose network analysis programs provide alternative measures, but available evidence for using one or another is largely anecdotal, involving illustrative analyses of one or another particular data set without reference to the diversity of network structures likely to be encountered in empirical research.

My purpose is to illustrate the extent to which different relation measures and pattern similarity measures can be expected to generate different structural equivalence results. Stated in another way, this is a note on the extent to which structural equivalence results are reliable across often used alternatives for measuring network relations and pattern similarity. To clarify later comparisons, I begin with a brief review of network relation measures and pattern similarity measures combined within popular structural equivalence measures.

[1] Still another is the algorithm used to aggregate and describe pattern similarity data. This is the third of the three key methodological decisions made in a structural equivalence analysis, but I leave it beyond the scope of this paper focused on relation and pattern similarity components in structural equivalence data. Fortunately, cluster analysis and multidimensional scaling methods of representing structural equivalence data are much more visible in published network analyses so variations between methods are more widely known and discussed.

2. Measuring relations

Given a network of binary choice data, path distances can be obtained to identify indirect connections within the network. The original choice data appear unchanged in the path distance data as the paths one step in length. Let z_{ij} represent the length of the network relation from i to j ($0 \leq z_{ij} \leq 1$) and let pd_{ij} be the corresponding path distance ranging from 1 (if j is one of i's sociometric choices) up to a empirical maximum of 1 less than the number of individuals in the network (if the network is a chain with i at the beginning and j at the end), and on to a theoretical value of infinity (if there is no chain of choices through which i can reach j). It is widely agreed that an individual's direct sociometric choices represent stronger relationships than his or her relations corresponding to long chains of indirect choices, but structural equivalence analyses by different individuals often use different functions to transform path distances into network relations. I distinguish three well known alternatives.

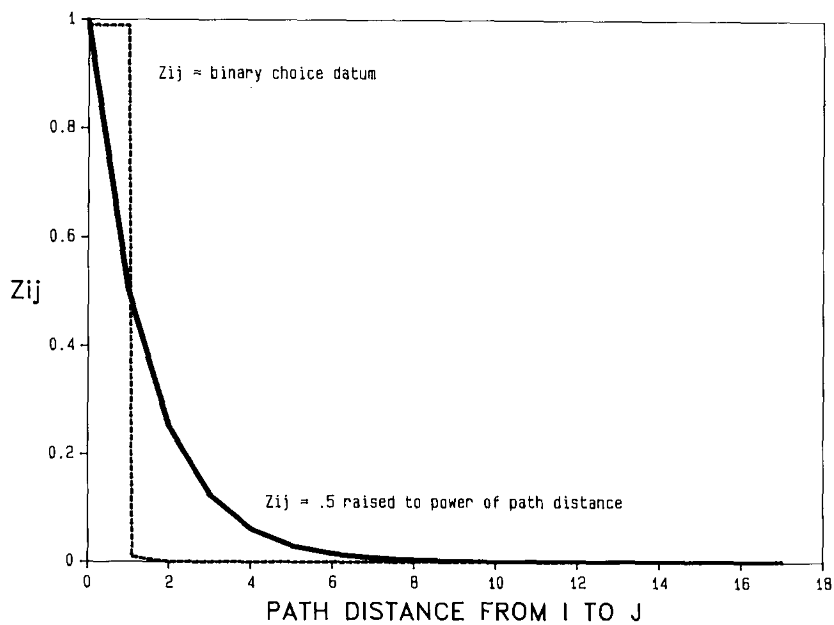

Fig. 1. Network relations (z_{ij}) from path distances (pd_{ij}).

The simplest is to ignore all indirect connections, treating the original binary choice data as direct measures of relationship. Such a function is described by the dashed line in Figure 1. Path distances are given on the horizontal axis and corresponding network relations are given on the vertical axis. This treatment of binary choice data as direct measures of network relations is common in analyses using CONCOR to construct blockmodels, although operationalizing structural equivalence with CONCOR is in no way limited to binary network data (*e.g.* White *et al.* 1976: 750, 759; Arabie and Boorman 1982). This idea is sometimes taken to the extreme of forcing quantitative measures of relationship into necessarily arbitrary categories of present and absent ties (*e.g.* Snyder and Kick 1979). However where categorical choice data have been obtained, using raw choice data as direct measures of network relations has the virtue of preserving the raw data obtained from respondents. The respondent named three people and those three people will be presumed to be the only individuals to whom the respondent sends relations. Even granting the arguable merit of such an argument, it does not extend to cover binary choice data created in ways never asserted by the respondents providing the original sociometric data. For example, White *et al.* (1976: 759) recode the Newcomb–Nordlie sociometric rank order data into two networks of binary relations; a network of liking where each person's top two rankings are assumed to be sociometric citations of attraction, and a network of antagonism relations where each respondent's bottom two ranks are assumed to be antagonism citations. In the same analytical style, Breiger (1976) recodes the data obtained with a seven point scale of professional contact into multiple networks of binary contact such as mutual contact, asymmetric awareness, and symmetric unawareness. These artful recordings are advanced on the argument that structural equivalence is better revealed with networks of contrasting relations (e.g. attraction versus antagonism). This argument, nicely summarized by Arabie and Boorman (1982), is a significant insight and compels emulation. Note however that the argument is in no way compromised by including indirect connections within each constructed network in the search for structural equivalence across networks. Moreover, there are advantages to doing so. Networks composed of only direct choices tend to be sparse, creating computational difficulties for models operationalizing structural equivalence with correlations, and lessening the reliability of structural equivalence measures more generally. Including indirect connections in the network data used to define structural equivalence typically fills in much of the empty space of the initial choice matrix. Further, indirect connections can improve the precision of structural equivalence measures. Individuals structurally equivalent in a matrix of path distances are equivalent with respect to their direct as well as indirect relationships. Further, choice data are often arbitrarily limited to the first two to five mentioned. Indirect connections such as friends of friends indicate less close contacts likely to have been cited if the study field work had recorded more sociometric choices.

To take advantage of the structural equivalence information provided by indirect connections, it is simple to measure relation strength as a constant function of path distance. Katz (1953) was one of the first to propose such a transformation by suggesting that the strength of a network relation be measured as a faction raised to the power of its correspondence path distance:

$$z_{ij} = \begin{cases} 1, & \text{if } i = j \\ [a]^{pd_{ij}}, & \text{if } i \text{ can reach } j \ (1 \le pd_{ij} \le N-1) \\ 0, & \text{if } i \text{ cannot reach } j \end{cases} \quad (1)$$

where the fraction a in Katz's numerical illustration was 0.5 and it continues to be set at 0.5, although Katz gave a more sophisticated definition of the constant in which $1/a$ is to be larger than the maximum eigenvalue of the choice matrix. The key point is that relations corresponding to long path distances are much weaker than relations corresponding to short ones, z_{ij} decreasing from 0.5 for direct choices, to 0.25 for two-step path distances, to 0.125 for three-step path distances, and so on. This transformation of path distances into network relations is described by the solid line in Figure 1. Note the sharp decay in relationship length. Relations between individuals connected by path distances of five or more choices ($z_{ij} = 0.5^5 = 0.016$) are nearly the same strength as relations between individuals completely unconnected with one another. The rate of decay with increasing path distance can be slowed by increasing the fraction a, but there is evidence to support the idea that relationship strength should decrease quickly beyond indirect connections through more than one intermediary (*e.g.* Friedkin 1984).

There are two virtues to such a function. The most important is that by bringing indirect connections into the network relations from which structural equivalence will be computed, it improves the precision and reliability of structural equivalence measures as discussed above. Second, it preserves the path distance information derived from the original choice data. You can tell immediately from the strength of a network relation how many choices were required to establish the relation (0.5 indicating a direct path, 0.25 indicating a two-step path, etc.).

Still, structural equivalence measures compare the similarity of relation patterns across individuals, and path distances of the same length can mean different things in different circumstances.

One drawback to a constant function such as the above is that it presumes that the number of sociometric choices elicited from each respondent corresponds to the number of the respondent's strongest relations. This is a problem in its own right, and a problem compounded by the practice of eliciting a fixed number of choices from each respondent (*e.g.* Holland and Leinhardt 1973; Hallinan 1974). Consider two respondents, one a member of a group of four close friends and the other a member of a group of six close friends. If each is asked to name their three best friends, then the first respondent has no problem because the number of elicited choices equals the boundary of his social circle of close friends. Two-step path distances will represent relations beyond his social circle. The second respondent can only name three of his five close friends and some two-step path distances from him will refer to his relations with uncited close friends through his cited friends back to friends he was unable to cite.

A second drawback to a constant function such as the above is that it presumes that relationship strength decays with increasing path distance at the same rate for all people. For example, if a respondent names three best friends who in turn each name three and none of the choices go to the same people, then the respondent has nine relations corresponding to two-step path distances. If another respondent names only one best friend who in turn names only one, then the respondent has only one two-step path distance relation within a small social circle of three people including himself. On the presumption that relations require energy to sustain, the second respondent's two-step path distance, reaching fewer people within a smaller social circle, indicates a stronger relationship than the many relations corresponding to two-step path distances from the first respondent.

A third alternative to measuring relation strength is to use a variable rate decay function based on path distance and aspects of the structural circumstances in which a path distance occurs. For example, Burt (1976: 118–119; 1982: 28–29) proposed the following function to prepare choice data for a structural equivalence analysis:

$$z_{ij} = \begin{cases} 1, & \text{if } i = j \\ 1 - f_{ij}/n_i, & \text{if } i \text{ can reach } j \ (1 \le pd_{ij} \le N - 1) \\ 0, & \text{if } i \text{ cannot reach } j \end{cases} \quad (2)$$

where n_i is the number of individuals that i can reach including himself in any number of choices and f_{ij} is the number of individuals that i can reach in the minimum number of choices needed to connect her with j. This is the default option in STRUCTURE for computing structural equivalence from siometric choice data so the function may be more widely used than it is known. Unless network relations are explicitly requested to equal input binary choice data, path distances will be derived from binary choice data and used in the above function to measure network relations. The argument for this transformation (and any other variable rate decay function of path distances) is that the rate at which the strength of a relation decreases with the increasing length of its corresponding path distance should vary with the social structure in which it occurs. Here, decay is a function of the number of people reached at each path distance compared with the total number reached at the boundary of an individual's social circle. The larger the group over which one has to distribute one's time and interpersonal energy, the weaker the relationship one can sustain with any one member of the group and the stronger the relations with people of relatively short path distance from you in the group.

Figure 2 illustrates the variation in relation strength possible around a given path distance in a network of 50 individuals. The extremes possible are indicated by dashed lines. If the network were a chain with the first person citing the second who cites the third and so on through the 49th person citing person 50, then the strength of relations for the first person would look like the dashed line at the top of Figure 2. Very few people are reached at each path distance, so it would be easy to sustain strong relations with individuals of path distance, two, three, or four steps away and these people are dramatically closer than the

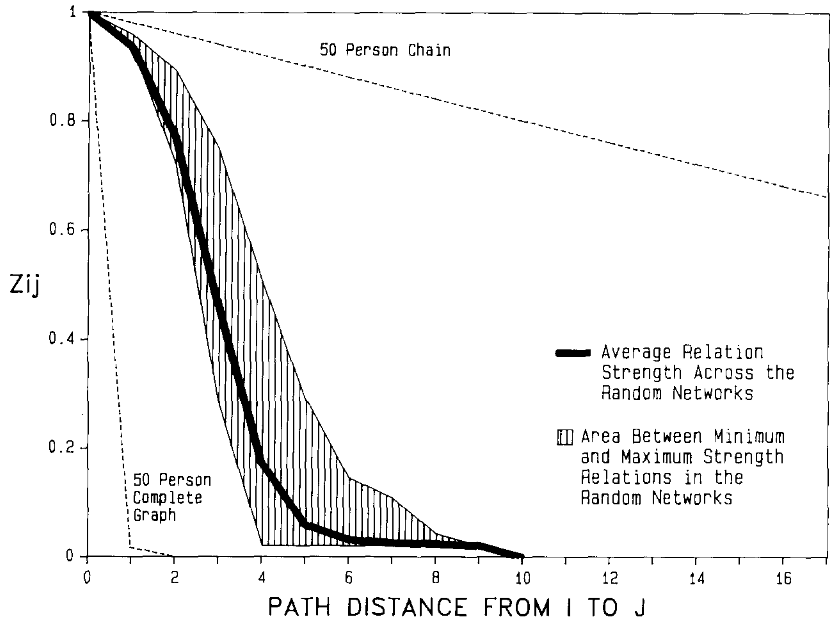

Fig. 2. Network relations (z_{ij}) from path distances (pd_{ij}) in 100 networks of uniform random choices among 50 people each making 3 choices.

49-step outer boundary of the social circle around the first person. In contrast, if the network were a complete graph with every person making the 49 citations required to connect him or her directly to every other person, then the strength of direct ties would be very weak given the difficulty of sustaining so many relationships at the same strength and the fact that direct ties define the outer boundary of each person's social circle. This is illustrated by the lowest dashed line in Figure 2.

Between these extremes in the graph are results more likely to be obtained in empirical research. The results describe 100 random networks of 50 people in which each person made 3 sociometric choices equally like to go to each other person. [2] No path distances were longer than 9 choices and the strength of relations decays quickly with path distance increasing beyond two-steps. The bold solid lines describes the mean values of z_{ij} for each path distance. The area of vertical strips

[2] Random sociometric choice data have been generated for this paper using the Monte Carlo options in STRUCTURE (Burt 1987) with a 639 seed number beginning each separate series of replicate networks.

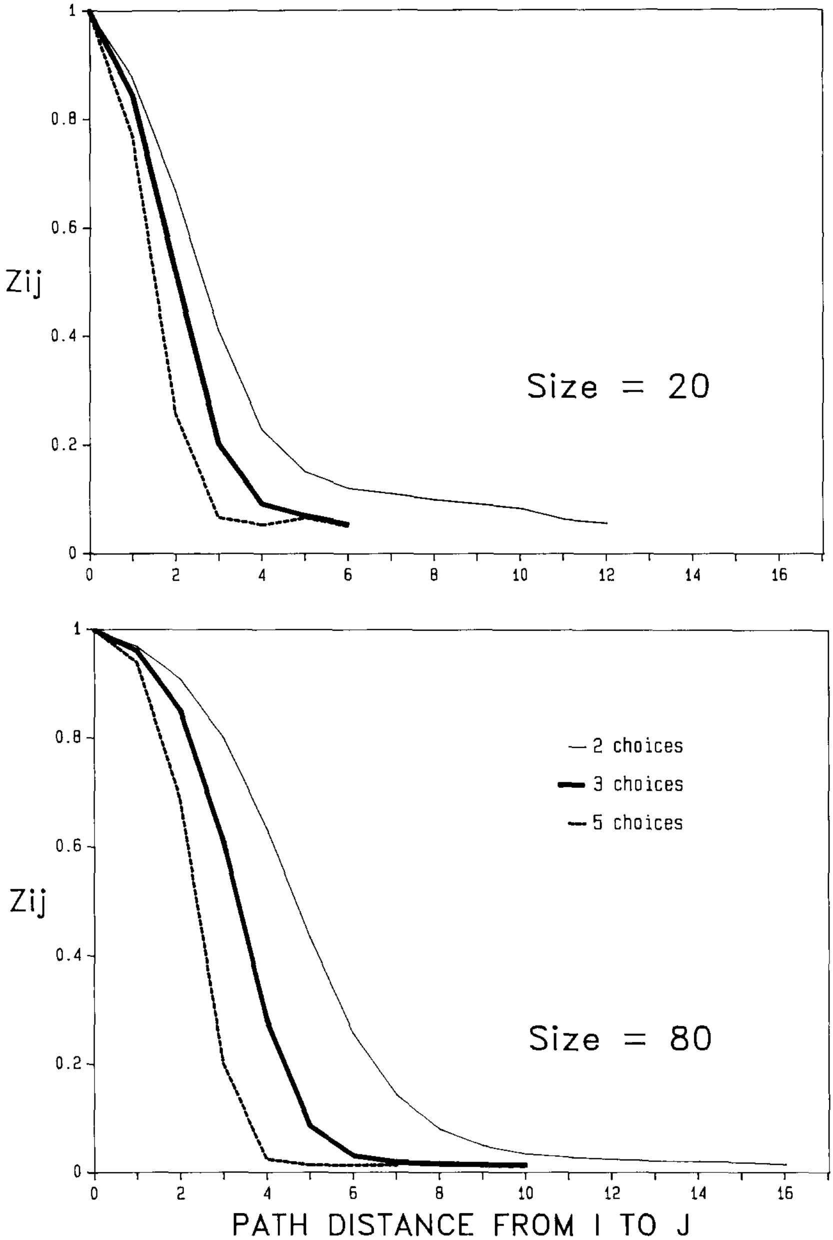

Fig. 3. Network relations (z_{ij}) from path distances (pd_{ij}) in 100 networks of uniform random choices (network size and number of choices varying as indicated).

indicates the variation in z_{ij} around each path distance length. For example, a four-step path distance represented a relationship of 0.510 strength for people reaching a small proportion of their contacts in four choices (top of the striped area) but it represented a much weaker 0.020 strength relationship for people who reached most of their contacts in fewer than four choices (bottom of the striped area).

Figure 3 illustrates the effect on the decay function of varying network size and number of sociometric choices. The results are based on relations in 100 networks with each respondent making the indicated number of sociometric choices drawn from a uniform random distribution across all other individuals in the network. As in Figures 1 and 2, path distance is given on the horizontal axis and the mean corresponding network relations is given on the vertical axis. A three choice limit is most popular (bold solid line in Figure 3) with few studies eliciting only two choices (light solid line in Figure 3) and few studies eliciting as many as five choices (bold dashed line in Figure 3). The rate at which relation strength decays with increasing path distance speeds up with the number of choices elicited (which increases the number of people reached with each length of path distance) and slows with increasing network size (which increases the likelihood of long path distances).

Here again, there are virtues and drawbacks. Like the constant function proposed by Katz, this measure has the virtues of incorporating indirect connections in the network data used to compute structural equivalence. Beyond the constant function, this measure adjusts the strength of relation implied by a path distance for the structural context in which the path distance occurs. However, the measure does this in an arbitrary way. There is no systematic evidence to support the function relating relation decay to the number of persons reached at each path distance; it is simply less obviously wrong than a constant decay function such as the one proposed by Katz and can draw some legitimacy from evidence on limits to the information that people can process simultaneously (*e.g.* Miller 1956).

3. Measuring pattern similarity

Lorrain and White's (1971) structural equivalence concept came to operational fruition in the mid-1970s with a flurry of articles proposing categorical models of structural equivalence, termed blockmodels (*e.g.* Breiger *et al.* 1975; White *et al.* 1976; Arabie and Boorman 1982), and continuous distance models (*e.g.* Burt 1976, 1977, 1982). The concept quickly evolved into more sophisticated, more intuitively appealing, more abstract models of equivalence (*e.g.* Sailer 1978; Winship and Mandel 1983; White and Reitz 1983), but structural equivalence remains the workhorse concept guiding substantive network studies of social structure, and the focus on my attention here.

Two individuals are structurally equivalent within a network to the extent that they have identical relations with every potential source and object of relations in the network. Measuring such a condition is a standard problem in the analysis of data profiles, with Cronbach and Gleser's (1953) review article still providing the classic statement of the profile level, scatter, and shape components in measures of profile similarity. The pattern, or profile, of relations defining individual j's network position can be arranged in a vector $\boldsymbol{Z}_j$ of $2N$ relation variables; N variables measuring j's relations to others ($z_{j1}, z_{j2}, \ldots z_{jN}$) and N variables measuring the relations received from others ($z_{j1}, z_{j2}, \ldots z_{Nj}$).[3] The extent to which individuals i and j are involved in identical relations so as to be structurally equivalent can be expressed as the Euclidean distance between their relation patterns (*cf.* Cronbach and Gleser 1953: 459):[4]

$$d_{ij(r)} = \left(\left[(\boldsymbol{Z}_j - \boldsymbol{Z}_i)'(\boldsymbol{Z}_j - \boldsymbol{Z}_i)\right]\right)^{1/2}, \tag{3}$$

where $'$ indicates transpose. This Euclidean distance between raw relation patterns is the default measure of structural equivalence in STRUCTURE. Dividing by the number of relations compared yields the root mean squared difference, the average difference in someone's relation with i versus his corresponding relation with j (these are the Euclidean distances generated by SYSTAT):

$$d_{ij(m)} = \left(\left[(\boldsymbol{Z}_j - \boldsymbol{Z}_i)'(\boldsymbol{Z}_j - \boldsymbol{Z}_i)\right] / [2N]\right)^{1/2},$$

[3] The extension to multiple networks is obvious and irrelevant to this note so equations are stated for structural equivalence within a single network. To get the equations for structural equivalence across K networks add a subscript k to each relation variable and sum relations across all K networks.

[4] These equations change slightly if diagonal elements in the choice matrix, relations with oneself, are arbitrary constants—as is typically the case with relations derived from sociometric choice data (*e.g.* see Burt 1987: 21).

which is merely the raw Euclidean distance divided by a constant, the square root of $2N$. The decision to use one or the other depends on the weight to be given to missing relations. The raw Euclidean distance measure ignores all relations beyond those involving i or j. The mean Euclidean distance averages differences between existing relations across as possible relations. Adding isolates to a network, for example, will decrease mean Euclidean distances but have no effect on raw Euclidean distances. This is a minor issue, of interest principally where metric distances will be compared across networks of different sizes.

Mean differences between patterns are a more serious concern. For example, input–output tables provide network relations between sectors of an economy. Relations are typically measured as the dollars of commodity sold by establishments in the row sector to establishments in the column sector. Economic and sociological theory about the structure of production relations, however, is not concerned with dollars of sales so much as they are concerned with the relative strength of relations. The raw input–output table data are usually divided by the column marginals of the table to produce network relations measuring the proportion of input to each column sector that is purchased from each row sector. In sociometric data, individuals can differ in their average tendencies to be involved in relations as a function of response bias or inaccurately measured relations, some seeing themselves as very active socially and being often cited while others report few contacts and themselves escape notice in other's citations. To control the effect of such differences on structural equivalence measures, the "level" component in relation patterns can be removed by subtracting out the mean strength of an individual's relations. Distance will be the root mean squared differences between relations adjusted for means (*c.f.* Cronbach and Gleser 1953: 460):

$$d_{ij(\mathrm{d})} = \left(\left[(Z_j - Z_i)'(Z_j - Z_i)\right]/[2N]\right)^{1/2}, \tag{4}$$

with vector elements defined as,

$$Z_j = \left(\boldsymbol{Z}_j - \bar{\boldsymbol{Z}}_j\right) = \{z_{jk} - \bar{z}_j\},$$

and $\bar{\boldsymbol{Z}}_j$ is a $2N$ vector of elements each equal to the mean, $\bar{z}_j$, of all relations involving person j (the mean for all elements in $\boldsymbol{Z}_j$). These distances can be computed as follows from the variances of i's and j's relations and the covariance between their relations:

$$d_{ij(\mathrm{d})} = \left(s_i^2 + s_j^2 - 2s_{ij}\right)^{1/2},$$

where s_j is the standard deviation of j's relations (the standard deviation of all elements in $\boldsymbol{Z}_j$, $s_j^2 = (\boldsymbol{Z}_j - \overline{\boldsymbol{Z}}_j)'(\boldsymbol{Z}_j - \overline{\boldsymbol{Z}}_j)/2N$) and s_{ij} is the covariance between j's and i's relations.

Further, again as a function of response bias or inaccurately measured relations, individuals can differ in the amplitude of their relationships, some individuals reporting relations that range from intensely close to intensely hostile while other individuals report little variation between their relationships. To control the effect of such differences on structural equivalence measures, the "scatter" component in relation patterns can be removed by dividing each relation by the standard deviation of an individual's relations. Distance will be the root mean squared difference between relations adjusted for means and standard deviations:

$$d_{ij(\mathrm{s})} = \left(\left[(\mathrm{Z}_j - \mathrm{Z}_i)'(\mathrm{Z}_j - \mathrm{Z}_i)\right]/[2N]\right)^{1/2}, \tag{5}$$

with vector elements defined as,

$$\mathrm{Z}_j = \left(\boldsymbol{Z}_j - \overline{\boldsymbol{Z}}_j\right)/s_j = \left\{(z_{jk} - \bar{z}_j)/s_j\right\}.$$

These distances could be computed as follows from the correlation (r_{ij}) between j's and i's relation patterns (*cf.* Cronbach and Gleser 1953: 461):

$$d_{ij(\mathrm{s})} = \left[2(1 - r_{ij})\right]^{1/2},$$

so, in other words:

$$r_{ij} = \left(1 - d_{ij(\mathrm{s})}^2\right)/2, \tag{6}$$

which makes it clear that the correlation between two relation patterns (the measure of structural equivalence in CONCOR) is proportional to the Euclidean distance between the two patterns stripped of their

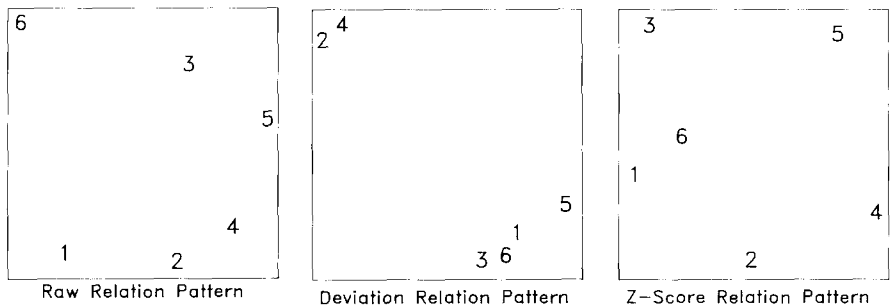

Fig. 4. Spatial maps of structural equivalence with and without level and dispersion components in relation patterns.

means and standard deviations. This is an important link between two ostensibly different pattern similarity measures often contrasted in debate over structural equivalence measures. The same structural equivalence results would be obtained from the clustering or multidimensional scaling algorithms typically used in network analysis if the r_{ij} or the $d_{ij(s)}$ were used to measure structural equivalence.

Very different structural equivalence results can be obtained with these measures. The verb should be emphasized; different results can be obtained, but different results will not always be obtained. The stronger the similarity between two relation patterns, the less it matters which of the alternatives is used to measure structural equivalence. I will return to this point in a moment.

Arguments for measuring structural equivalence as similarity between raw relation patterns do not rule out the value of holding level and scatter constant in some data sets. Rather, they inveigh against the indiscriminate use of such controls because relation pattern level and scatter can be significant features of social structure in a study population (*e.g.* Burt 1982: 47–48, 1986; Burt and Minor 1983: 274–277). [5] The potentially different results possible with these measures are illustrated in Figure 4 with three spatial maps of structural equivalence

[5] The one published argument for holding level and scatter constant—among the many analyses in which they are merely held constant by default without an explanation for holding them constant—argues that variation in the margins of a network is not structural information and so should be held constant before beginning the search for structural equivalence (Faust and Romney 1985). This assertion seems to me just too sweeping for the the variety of ways in which relations are measured in empirical research.

measures computed for the following hypothetical network (where integer values of relations are used to highlight the point being illustrated):

–	3	3	3	3	0
3	–	0	4	2	0
3	0	–	0	1	0
3	4	0	–	0	0
3	2	1	0	–	0
0	0	0	0	0	–

The spatial maps are routine STRUCTURE output based on the first and second eigenvectors for the distance matrix. Structurally equivalent individuals are close together in the map. In the first map, structural equivalence is computed from patterns of raw relations (Equation 3). Note that the central figure, person 1, is far away from the network isolate, person 6. Very different relations define their positions in the network. But the principal difference between the central figure and the isolate is the mean strength of their relations with others. When levels are removed from relation patterns, the center and periphery of the network are folded in upon one another. In the second and third maps in Figure 4, the leader and the isolate are right next to one another.

The alternative pattern similarity measures are all readily available (*e.g.* as distance options in STRUCTURE), but in large part because of default options in readily available computer programs, structural equivalence is most often measured as similarity in all aspects of relation pattern (*e.g.* Euclidean distance between raw relation patterns in STRUCTURE) or with level and scatter held constant within relation patterns (*e.g.* correlation between relation patterns in CONCOR, *ie.* Euclidean distance between *z*-score relation patterns in STRUCTURE). Comparisons across studies have been further made difficult by specific pattern similarity measures being used typically with a single kind of relation measure, CONCOR being used typically to measure structural equivalence from *z*-score patterns of binary network relations and STRUCTURE being used typically to measure structural equivalence from raw patterns of network relations based on path distances normalized by the above described frequency decay function.

4. Study design

Combining the three relation measures with the two most different and most often used measures of pattern similarity defines six structural equivalence variables in the factorial design given in Table 1. The first is the matrix of raw Euclidean distances (Equation 3) between patterns of binary relations. A vector, $D1$, contains the $N(N-1)/2$ distances below the diagonal of the N by N distance matrix. The second variable, $D2$, contains Euclidean distances between raw patterns of network relations based on path distances normalized by the constant decay function (Equation 1), and the third variable, $D3$, contains Euclidean distances between raw patterns of network relations based on path distances normalized by the variable rate decay function determined by the number of individuals reached at each path distance (Equation 2). The fourth, fifth, and sixth variables ($D4$, $D5$, and $D6$) contain Euclidean distances between patterns of z-score network rela-

Table 1
Factorial design generating structural equivalence measures

	Structural equivalence measure					
	$D1$	$D2$	$D3$	$D4$	$D5$	$D6$
Relation measure						
Binary choice data	×	.	.	×	.	.
Constant decay function based on path distance	.	×	.	.	×	.
Frequency decay function based on path distance	.	.	×	.	.	×
Pattern similarity measure						
Raw relation pattern	×	×	×	.	.	.
Z-score relation pattern	.	.	.	×	×	×

Note Each of the six structural equivalence measures is a combination of a relation measure and a pattern similarity measure as indicated by the ×s. For example, $D1$ contains the 1770 elements in lower diagonal portion of the distance matrix computed by comparing patterns of raw binary relations. The constant rate decay function is given in Eq. (1) and the frequency decay function is given in Eq. (2). The Euclidean distance based on comparing raw relation patterns is given in Eq. (3) and distance based on comparing z-score relation patterns is given in Eq. (5) with its transformation into a correlation coefficient given in Eq. (6).

tions (Equation 5)—that is to say correlations (Equation 6)—respectively based on the three alternative relation measures.

This study design thus covers the domain of most often obtained structural equivalence measures, ranging from correlations between patterns of binary network relations (CONCOR), through many never used alternatives, to Euclidean distances between raw patterns of network relations based on path distances normalized by the frequency decay function (STRUCTURE). I have not included distance between patterns of deviation score relations because this alternative has never to my knowledge been used by itself and it lies between the included alternatives of raw relation patterns at one extreme and z-score relation patterns at the other extreme.

I am looking for components in the covariation between the alternative structural equivalence measures. When structural equivalence in a study population is detected similarly by each of these measures, the correlation matrix among the six variables will have a rank of one, each variable $D1$ through $D6$ similarly measuring the relative extent to which pairs of individuals are structurally equivalent. Alternatively, when correlation measures of structural equivalence generate results different from measures based on raw relation patterns—regardless of the manner in which relations are measured—then the correlations among $D1$, $D2$, and $D3$ and the correlations among $D4$, $D5$, and $D6$ will be stronger within each group than between the groups. When structural equivalence based on binary network relations is distinct from structural equivalence based on path distances—regardless of whether patterns of raw relations or patterns of z-score relations are compared—then $D1$ and $D4$ will be more strongly correlated with one another than either is with other measures.

I will look for these covariance components in a Monte Carlo network analysis of four study populations. The study populations have been designed to range from the extreme of random relations through the other extreme of social structure defined by structural equivalence under a strong criterion. With structural equivalence increasingly clear across these study populations, comparisons across the study populations should show increasing similarity between the alternative structural equivalence measures.

Using the Monte Carlo options in STRUCTURE, I drew 100 replicate networks in each study populations of 50 people, recording three sociometric choices from each respondent, prohibiting self cita-

Table 2
Density tables for four study populations

Uniform random	0.061	0.061	0.060
	0.059	0.062	0.059
	0.059	0.059	0.062
Center–periphery	0.024	0.087	0.048
	0.023	0.091	0.048
	0.022	0.091	0.047
Multiple status hierarchy	0.295	0	0
	0.100	0.100	0
	0	0.075	0.075
Strong equivalence multiple status hierarchy	1.00	0	0
	1.00	1.00	0
	0	1.00	1.00

Note: Each density table is based on 100 networks of binary choices among 50 people with persons 1 through 10 assigned to the first position, persons 11 through 30 assigned to the second position, and persons 31 through 50 assigned to the third position. Choices were generated at random from distributions discussed in the text.

tions. [6] Each run producing one of the structural equivalence measures ($D1$, $D2$, $D3$, $D4$, $D5$, or $D6$) began with the same random seed number, 639, so each of the alternative measures is based on the same initial 100 networks of sociometric choice data. Density tables for each study population are presented in Table 2. The first 10 people are assigned to position one, the second 20 are assigned to position two,

[6] Three is an upper limit like the limit imposed by fixed choice sociometric questionnaire items. Like the fixed choice sociometric item, this limits the variation in the choice matrix row marginals to focus on variation in the column marginals. Also like the data generated by fixed choice sociometric items, some variation in number of choices is possible from one row to the next. If a row person cited persons 3, 4, and 4—citing person 4 twice—only two choices were recorded from the row. In practice, with the ratio of choices to network size so small, most respondents made three separate choices. This can be seen from the magnitudes of the cell densities in Table 2 for the uniform random networks. However, where choices are constrained to a small number of people receiving choices, the odds of row individuals making three different choices go down. The most extreme example here are the choices made in the third study population among the 10 occupants of the top position in the multiple status hierarchy. Their choices were limited to other leaders, so the odds of any 2 choices going to any one of the 9 potential recipients are relatively good. The 0.295 density of ties among them reported in Table 2 means that each occupant cited an average of 2.7 different people rather than the 3 possible.

and the remaining 20 are assigned to position three. Position assignments in the first two study populations are merely for illustration. [7]

The first study population has no social structure. Each of the 50 respondents made three sociometric choices equally likely to go to each other person in the network. With 3 choices distributed among 49 people, each person has a 0.061 probability of being cited by any one other person. Note in Table 2 that the cells of the density table averaged across the 100 uniform random networks are all about the same and close to this expected value. Social structure in this study population is no more than random noise.

The second population has a center–periphery structure. Again 100 networks were drawn in which each of the 50 respondents made three sociometric choices, but here the choices were drawn from a normal probability distribution. In other words, people in the middle of the network were more likely to receive choices than people at the beginning or end of the network. Note in Table 2 that the cells in the middle column of the center–periphery density table are higher than the cells in the first or third columns. The cells in the third column are higher than those in the first column because the first is occupied by 10 people at the far left of the probability distribution. The 20 occupants of the third position are distributed closer to the center of the probability distribution and so more likely to be cited. There is structure here, unlike the first study population, but it is contained entirely in the marginals of the choice matrix. This is similar to the structure of the HAMS network used by Bernard *et al.* (1980), Burt and Bittner (1981), Faust and Romney (1985), and Burt (1986) to compare structural equivalence results obtained from raw relation patterns with results obtained from z-score relation patterns.

The third study population is a multiple status hierarchy. Structure is evident in the marginals and within the choice matrix. The 100 networks were drawn for 50 individuals making three choices equally

[7] This is a fitting point to note that generating these data is a simple matter with suitable hardware. On a 12 MHz IBM compatible microcomputer writing extensive data output to hard disk, STRUCTURE required 50 minutes to generate the 122,500 scores in $D1$ and a noticeably longer 195 minutes to derive path distances, normalize relations, and standardize relation patterns for the 122,500 scores in $D6$. In all, with merging and editing, a microcomputer worked for a couple of days generating the structural equivalence data on each study population. Ample scratch space on a mass storage device is essential. Slightly less than 15 megabytes of data were produced for each of the first three study populations.

likely to go to each other person in the network. However, the range of the distribution was constrained differently for different statuses by constraining random choices to nonzero cells in a target image matrix (given by the density table for the fourth study population in Table 2). The 10 leaders at the top of the hierarchy were constrained to choose one another, so there is a high density of choices among them. The 20 brokers in position two could chose one another or any of the leaders in position one. The densities in the second row for this study population are lower than the 0.265 in the first row because each person's three choices were spread over three times as many people (30 rather than 10). The remaining 20 people, at the bottom of the hierarchy, could chose one another or any of the brokers in position two and so spread their relations over even more people (40), generating the lowest densities in the table for this study population.

The fourth study population has the same structure as the third except that nonzero densities now equal their maximum value. Every occupant of a position cites every other occupant of his position. Every broker cites every leader. Every occupant of the bottom position cites every broker. The social boundaries around structurally equivalent people in this population are extremely clear and so should be recovered by almost any structural equivalence measure.

5. Results

There are 122,500 scores on each of the structural equivalence variables in each study population (1,225 distances between pairs of 50 people within each of 100 networks). [8] Since structural equivalence under anything less than a strong criterion is defined in relative terms—two relation patterns being more similar than specific other pairs of patterns—I analyze correlations among the six structural equivalence variables rather than covariances or sums of squares. The correlation matrix among the six structural equivalence measures within each study population has been disaggregated into its six principal components. The eigenvalue defining each component is plotted in Figure 5, increas-

[8] Note, however, that each of the networks drawn from the fourth study population is identical to every other network from the population since all allowed choices have been made in every network.

ing from left to right within each graph. Dividing these magnitudes by six, the total variance in the correlation matrix, yields the ratio often presented as the proportion of variance described by a principal component.

There is substantial reliability across alternative structural equivalence measures. Variance is concentrated in the maximum eigenvalue to the right of each graph, indicating that the relation and pattern similarity measures generate very similar, reliable, structural equivalence results. This is minimally so in the uniform random network (to

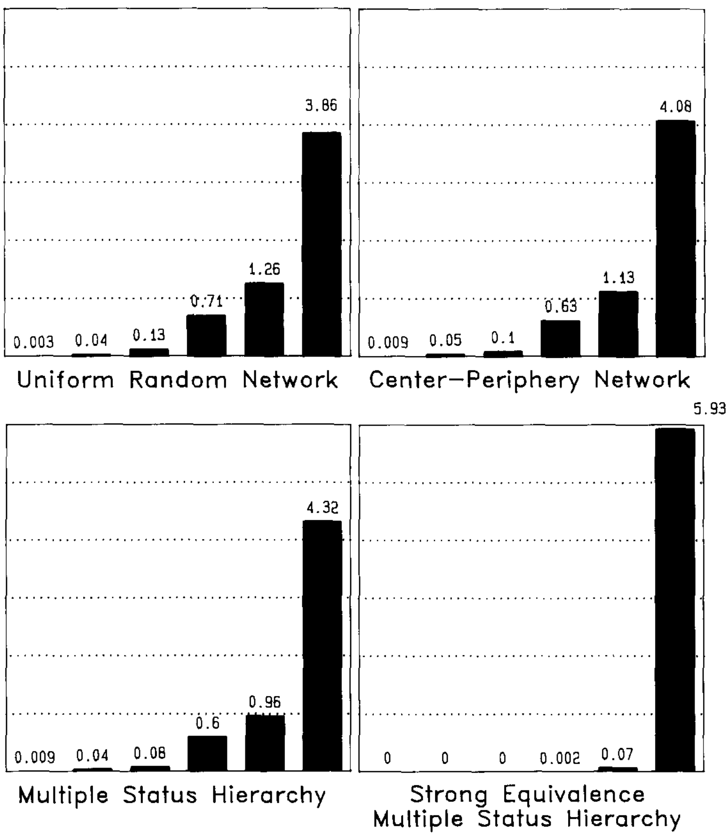

Fig. 5. The reliability of structural equivalence results increases with the clarity of status boundaries in social structure (eigenvalues of the correlation matrix among six measures of structural equivalence within each study population).

the upper left of Figure 5). But even there, where network structure is no more than random noise, a single principal component describes 64 percent of the variance in the six structural equivalence measures. Where status boundaries are defined by strong equivalence, the alternative measures yield results virtually identical across measures (to the lower right of Figure 5). A single principal component describes 99 percent of the alternative measure variance in the strong equivalence multiple status hierarchy.

These results are encouraging, but unrealistic by some unknown amount. In these principal component results, structural equivalence reliability is not distinguished from method covariance. This point can be illustrated with the covariance structure diagrammed in Figure 6. The factorial design defines five methods factors potentially generating covariation between the six structural equivalence measures. Measures *D*1, *D*2 and *D*3 are all based on similarity between raw relation patterns. Measures *D*4, *D*5 and *D*6 are all based on similarity between *z*-score relation patterns. Measures *D*1 and *D*4 are both based on treating binary choice data as direct measures of relations. Measures *D*2 and *D*5 are both based on the constant decay function of path distance and measures *D*3 and *D*6 are both based on the frequency decay function of path distance. The remaining covariation between the six structural equivalence measures, captured in the $\mathbf{D}_{ij}$ variable in Figure 6, measures reliability with methods factors held constant across

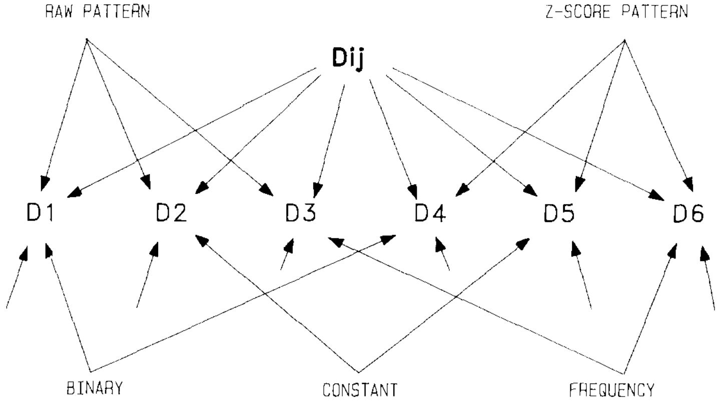

Fig. 6. Components in structural equivalence reliability across measures of network relations and pattern similarity.

the structural equivalence measures. More specifically, with all factor loadings set to 1.0, the standard deviation of $\mathbf{D}_{ij}$ will vary from 0 to 1 with the reliability of structural equivalence results within each study population. Least squares estimates of the $\mathbf{D}_{ij}$ standard deviation are 0.691 in the uniform random structure, 0.693 in the center–periphery structure, 0.736 in the multiple status hierarchy, and 0.990 in the strong equivalence multiple status hierarchy. [9] These reliabilities are high and increase (with increasingly clear status boundaries) across the study populations in a manner similar to the increasing eigenvalues in Figure 5. This point is clear in Figure 7. The estimates of reliable variance adjusted for methods factors (variance of $\mathbf{D}_{ij}$ in Figure 6) are plotted across the study populations with the principal component estimates of reliable variance (ratio of maximum eigenvalue in Figure 5 to total variance).

Before becoming sanguine about these results, note two points: Reliability increases in Figure 7 only slightly from the completely random networks to the center–periphery networks, to the network drawn from a multiple status hierarchy. The major increase occurs in the fourth study population where status boundaries are defined by strong structural equivalence. Further, a single principal component dominates the correlation matrix among alternative measures only in the fourth study population. In this study population, the relation patterns of all individuals occupying a status are identical. Such a condition is rare in empirical research, and below this extreme clarity

[9] The estimates were obtained numerically with the least squares algorithm in LINCS (Schoenberg 1987) for a factor analysis model in which the factor loading matrix was completely constrained to the following pattern:

1	1	0	1	0	0
1	1	0	0	1	0
1	1	0	0	0	1
1	0	1	1	0	0
1	0	1	0	1	0
1	0	1	0	0	1

where rows refer to the six structural equivalence measures and columns define the seven factors. Residual variance in the six observed measures was unconstrained. Variance in the reliability factor ($\mathbf{D}_{ij}$) was unconstrained, and variance of the six methods factors was unconstrained. The model contains 12 unknown variances to be estimated from 15 observed moments. Each unknown variance is identified. The high reliability of the measures resulted in small negative estimates for four of the 24 residual variances estimated across the four study populations. Negative variance estimates were constrained to a value of 0.0001 and parameters were re-estimated.

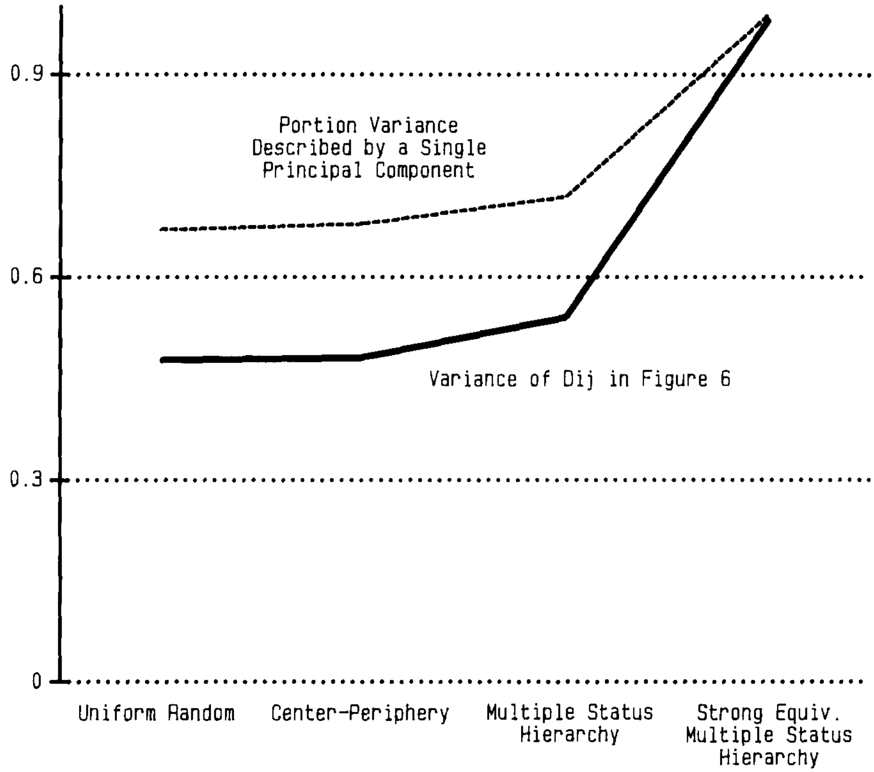

Fig. 7. Aggregate reliability (Fig. 5) and reliability adjusted for methods factors (Fig. 6) increase similarly across the study populations.

of status boundaries, there is substantial methods variation in structural equivalence results. This is illustrated in Figure 5 by the magnitudes of the second principal components. It is illustrated in Figure 7 by the solid line for adjusted reliability being much lower than the dashed line describing reliability with structural equivalence and methods factors confounded. The two lines in Figure 7 only meet for the strong equivalence hierarchy of the fourth study population where there is virtually no methods covariance between the alternative structural equivalence measures.

Figure 8 shows how the alternative measures contribute to the second dimensions within each study population. The six measures are positioned in the graph for each study population according to their loading on the first principal component (horizontal axis, measuring structural equivalence reflected in all the measures) and their loading on the second principal component (vertical axis, measuring an amalgam of the methods factors in Figure 6). The dashed lines in each graph cross at the zero point on both axes. The large first eigenvalue in the correlation matrix for each study population shows up in Figure 8 with all six structural equivalence measures loading positively on the first principal component (far right on the horizontal axis). The similar results obtained with any of the measures in the fourth study population shows up in the graph at the upper left corner of Figure 8 with all six measures appearing next to one another at the top of the first dimension with almost no variation in the second dimension.

The new information communicated in Figure 8 concerns the relative importance of relation versus pattern similarity measures for differences between the structural equivalence measures. Three points in particular are illustrated. First, note that structural equivalence

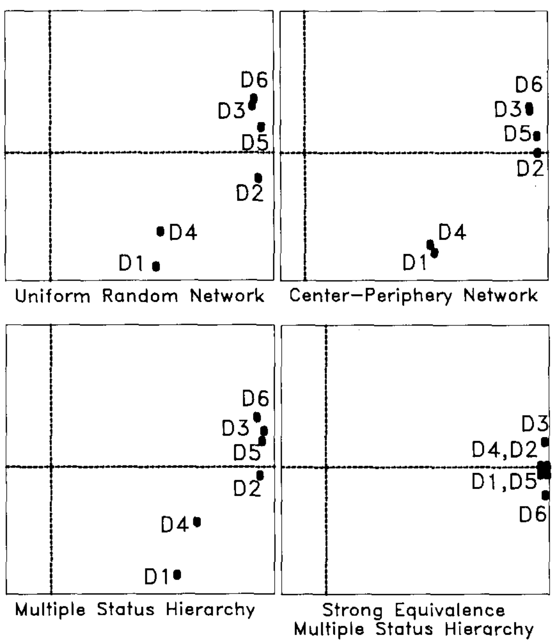

Fig. 8. Relation measures contribute more than pattern similarity measures to the unreliability of structural equivalence results (loadings of each structural equivalence measure on the first two principal components within each study population).

variables based on the same relation measure are closer together than are variables based on the same pattern similarity measure; *D*1 is always close to *D*4, *D*2 is always close to *D*5, and *D*3 is always close to *D*6. In other words, the principal differences between the alternative structural equivalence measures stem not from differences between pattern similarity measures (raw relation patterns versus correlated relation patterns) but rather from the differences between relation measures. Second, note that the most deviant results are obtained when binary choice data are treated as direct measures of network relations. To the extent that the six structural equivalence variables differ in a population, *D*1 and *D*4 differ most from the other measures. The difference between the constant decay function (*D*2 and *D*5) and the frequency decay function (*D*3 and *D*6) is discernible, but negligible in comparison to the difference between any of the four measures and the two based directly on binary choice data. Note also that when *D*1 and *D*4 are most different from the other measures, *D*4 is always closer to the structural equivalence results obtained with path distance measures of relationship. This is at least a comforting note for the early structural equivalence analyses based on correlated patterns of binary data. In these data, more reliable structural equivalence results are obtained by correlating binary choice data than by computing Euclidean distances between raw patterns of binary choice data.

6. Conclusions

I draw three conclusions from these results. First, there is significant reliability in structural equivalence results across alternative measures. Second, this reliability increases with the clarity of boundaries between statuses in a study population. Third, the noticeable differences between structural equivalence measures that exist under conditions at all weaker than strong equivalence are principally a function of how relations are measured rather than how relation pattern similarities are measured. The differences between structural equivalence measured as similarity between raw relation patterns versus similarity (correlation) between *z*-score relation pattern are trivial in comparison to the differences produced by treating binary choice data as a direct measure of relationship rather than measuring relations to take into account indirect connections through multiple step path distances. I stress once again that these results concern measures of structural equivalence—measures of the extent to which two relation patterns are identical. The results lead to no conclusions about the more abstract equivalence models that have been proposed in papers by Mandel, Reitz, Sailer, White, and Winship. The results presented here do raise interesting questions about the magnitude of differences among these models relative to differences in measuring relations, but in this paper I have focused on measures of structural equivalence, the class of equivalence measures typically used in contemporary empirical research to operationalize network positions.

The inference for applied network analysis is that (a) structural equivalence should be computed from path distance measures of network relations (however normalized) rather than being computed directly from patterns of binary choice data, and (b) renewed methodological attention should shift from how we measure pattern similarity to how we measure relationships.

References

Arabie, P. and S.A. Boorman
1982 "Blockmodels: developments and prospects." in: H.C. Hudson (ed.) *Classifying Social Data*. San Francisco: Jossey-Bass.

Bernard, H.R., P.D. Killworth and L.D. Sailer
1980 "Informant accuracy in social network data IV: a comparison a clique-level structure in behavioral and cognitive network data." *Social Networks 2*: 191–218.

Breiger, R.L.
1976 "Career attributes and network structure: a blockmodel study of a biomedical research speciality." *American Sociological Review 41*: 117–135.

Breiger, R.L., S.A. Boorman and P. Arabie
1975 "An algorithm for clustering relational data with application to social network analysis and comparison with multidimensional scaling." *Journal of Mathematical Psychology 12*: 328–383.

Burt, R.S.
1976 "Positions in networks." *Social Forces 55*: 93–122.

Burt, R.S.
1977 "Positions in multiple network systems, part one." *Social Forces 56*: 106–131.

Burt, R.S.
1982 *Toward a Structural Theory of Action*. New York: Academic Press.

Burt, R.S.
1986 "A cautionary note." *Social Networks 8*: 205–211.

Burt, R.S.
1987 STRUCTURE. Version 3.2, Technical Report #TR2, Center for the Social Sciences, Columbia University.

Burt, R.S. and W.M. Bittner
1981 "A note on inferences regarding network subgroups." *Social Networks 3*: 71–88.
Burt, R.S. and M.J. Minor (eds.)
1983 *Applied Network Analysis*. Beverly Hills: Sage.
Cronbach, L.J. and G.C. Gleser
1953 "Assessing similarity between profiles." *Psychological Bulletin 50*: 456–473.
Faust, K. and A.K. Romney
1985 "Does STRUCTURE find structure? A critique of Burt's use of distance as a measure of structural equivalence." *Social Networks 7*: 77–103.
Friedkin, N.
1984 "Structural cohesion and equivalence explanations of social homogeneity." *Sociological Methods & Research 12*: 235–261.
Hallinan, M.T.
1974 *The Structure of Positive Sentiment*. New York: Elsevier.
Holland, P.W. and S. Leinhardt
1973 "The structural implications of measurement error in sociometry." *Journal of Mathematical Sociology 3*: 85–111.
Katz, L.
1953 "A new status index derived from sociometric analysis." *Psychometrika 18*: 39–43.
Lorrain, F.P. and H.C. White
1971 "Structural equivalence of individuals in social networks." *Journal of Mathematical Sociology 1*: 49–80.
Miller, G.S.
1956 "The magical number seven, plus or minus two: some limits on our capacity for processing information." *Psychological Review 63*: 81–97.
Sailer, L.D.
1978 "Structural equivalence: meaning and definition, computation and application." *Social Networks 1*: 73–90.
Schoenberg, R.J.
1987 LINCS. Kensington, MD: RJS Software.
Snyder, D. and E.L. Kick
1979 "Structural position in the world system and economic growth." *American Journal of Sociology 84*: 1096–1126.
White, D.R. and K.P. Reitz
1983 "Graph and semigroup homomorphisms on networks of relations." *Social Networks 5*: 193–234.
White, H.C., S.A. Boorman and R.L. Breiger
1976 "Social structure from multiple networks, I. blockmodels of roles and positions." *American Journal of Sociology 81*: 730–780.
Winship, C. and M. Mandel
1983 "Roles and positions: a clique and extension of the blockmodeling approach," in: S. Leinhardt (ed.) *Sociological Methodology 1983/84*. San Francisco: Jossey-Bass.